

Caustic avoidance in Hořava-Lifshitz gravity

Shinji Mukohyama

Institute for the Physics and Mathematics of the Universe (IPMU)

The University of Tokyo

5-1-5 Kashiwanoha, Kashiwa, Chiba 277-8582, Japan

Abstract

There are at least four versions of Hořava-Lifshitz gravity in the literature. We consider the version without the detailed balance condition with the projectability condition and address one aspect of the theory: avoidance of caustics for constant time hypersurfaces. We show that there is no caustic with plane symmetry in the absence of matter source if $\lambda \neq 1$. If $\lambda = 1$ is a stable IR fixed point of the renormalization group flow then λ is expected to deviate from 1 near would-be caustics, where the extrinsic curvature increases and high-energy corrections become important. Therefore, the absence of caustics with $\lambda \neq 1$ implies that caustics cannot form with this symmetry in the absence of matter source. We argue that inclusion of matter source will not change the conclusion. We also argue that caustics with codimension higher than one will not form because of repulsive gravity generated by nonlinear higher curvature terms. These arguments support our conjecture that there is no caustic for constant time hypersurfaces. Finally, we discuss implications to the recently proposed scenario of “dark matter as integration constant”.

1 Introduction

Hořava recently proposed a power-counting renormalizable theory of gravitation [1, 2]. Since one of the most important aspects of the theory is a Lifshitz-type anisotropic scaling, it is often called Hořava-Lifshitz gravity. Various aspects of this theory were investigated [3]-[59].

Actually, in the literature there are at least four versions of the theory: with/without the detailed balance condition; and with/without the projectability condition. Among them, only the version without the detailed balance condition with the projectability condition has a potential to be theoretically consistent and cosmologically viable while there still are many unsolved issues. Hořava's original proposal [1] was with the projectability condition and with/without the detailed balance condition ¹.

The detailed balance condition restricts the form of potential in the 4-dimensional Lorentzian action to a specific form in terms of a 3-dimensional Euclidean theory. From cosmological viewpoint, this condition leads to obstacles [12, 42] and thus must be abandoned.

On the other hand, the projectability condition stems from the fundamental symmetry of the theory, i.e. the foliation-preserving diffeomorphism invariance, and thus must be respected. The foliation-preserving diffeomorphism consists of the 3-dimensional spatial diffeomorphism and the space-independent time reparametrization. Since the lapse function is essentially the gauge degree of freedom associated with the time reparametrization, it is natural to restrict it to be space-independent. This is the projectability condition, and one can easily show that the foliation-preserving diffeomorphism preserves this condition. The projectability condition implies that the Hamiltonian constraint is not a local equation satisfied at each spatial point but an equation integrated over a whole space. Abandoning the projectability condition and imposing a local version of the Hamiltonian constraint would result in phenomenological obstacles [36] and theoretical inconsistencies [37]. Those problems disappear once the projectability condition is respected and if only the global Hamiltonian constraint is imposed (see, for example, section 5 of [41]). Note that Hořava's original proposal was with the projectability condition.

For these reasons, in the present paper we restrict our attention to the version without the detailed balance condition with the projectability condition. We suppose that the dynamical critical exponent z in the ultraviolet (UV) is equal to or larger than 3 since power-counting (super-)renormalizability requires it. To the best of the

¹Hořava put much emphasis on the detailed balance condition but considered it as just a way to reduce the number of independent coupling constants.

author’s knowledge, no explicit inconsistency has been found against this version ².

This version of Hořava-Lifshitz gravity, with general z , has an interesting cosmological consequence even in the infrared (IR). The global Hamiltonian constraint is less restrictive than its local version, and allows a richer set of solutions than in general relativity. Actually, a component which behaves like pressureless dust emerges as an “integration constant” of dynamical equations and momentum constraint equations [41]. As a result, classical solutions to the infrared limit of Hořava-Lifshitz gravity can mimic solutions to general relativity plus cold dark matter. We shall discuss more about the “dark matter” in Sec. 4.

Also in the UV, cosmology based on Hořava-Lifshitz gravity has number of interesting properties. The anisotropic scaling with the dynamical critical exponent $z = 3$ leads to a new mechanism of generating scale-invariant cosmological perturbations [8]. The nonlinear higher spatial curvature terms lead to regular bounce solutions in the early universe [4, 9]. It was also suggested that nonlinear higher spatial curvature terms might make the flatness problem milder than in general relativity [5]. Since these mechanisms do not rely on the detailed balance condition or a local version of the Hamiltonian constraint, they can be applied to the version we are considering now.

Now, while cosmological consequences are interesting and it is worthwhile exploring further, one has to be aware that there are many fundamental issues to be addressed in the future. First of all, renormalizability beyond power-counting has not been proven. Second, the renormalization group (RG) flow of various coupling constants has not been investigated. In particular, recovery of general relativity in the IR relies on the assumption (or hope) that the parameter λ (see the next section) flows to 1 in the IR. Without knowing the condition for this behavior to be realized, we cannot be sure about recovery of general relativity in the IR. Third, this theory has not yet been intended to be a part of unified theory. Clearly, further developments or/and embedding into a “bigger” theory (or other way around) is needed. In particular, since the “limit of speed” is subject to the RG flow [57], we need a new idea to ensure that different species including those in the standard model of particle physics are somehow related to each other so that their “limits of speed” agree with the “velocity of light” within experimental limits ³. Fourth, an analogue of the

²Ref. [55] argued that the version with the projectability condition is also inconsistent, based on the following two claims: (i) constant-time hypersurfaces form caustics; (ii) the scalar graviton gets strongly coupled at all scales in Minkowski background even away from $\lambda = 1$. Actually, these claims are not correct. See the last paragraph of subsection 3.1, and the fourth-to-the-last and third-to-the-last paragraphs of section 4 of the present paper.

³See e.g. refs. [60, 61, 62] for tight experimental limits on Lorentz violation.

Vainshtein effect [63] must be investigated for the scalar graviton in the limit $\lambda \rightarrow 1$. Unlike general relativity, Hořava-Lifshitz gravity has not only a tensor graviton with two polarizations but also a scalar graviton. The dynamics of the scalar graviton and its fate in the IR must be investigated in details. Since the time kinetic term (together with gradient terms) of the scalar graviton vanishes in the $\lambda \rightarrow 1$ limit [1, 2], one has to take into account nonlinear interactions to see if it really decouples.

Aside from those issues, there is another important question. In general relativity, flow of pressureless dust generically forms caustics. Thus, one might expect that the flow of “dark matter as integration constant” would also form caustics. If this were the case then constant time hypersurfaces would develop singularities since the flow of “dark matter” is orthogonal to constant time hypersurfaces. This would be a disaster for Hořava-Lifshitz gravity with the projectability condition since the constant time hypersurfaces have physical meaning in this theory. Indeed, if caustics formed then the extrinsic curvature would diverge.

In this paper, we shall argue that this naive expectation is not correct and conjecture that there is no caustic for constant time hypersurfaces. While the proof of this conjecture is beyond the scope of this paper, we shall provide supporting arguments. In particular, we shall show that there is no caustic with plane symmetry in the absence of matter source if $\lambda \neq 1$. Since near would-be caustics the system enters the UV regime and λ is expected to deviate from 1 via RG flow ⁴, this implies that caustics with codimension one do not form in the absence of matter source. We shall argue that inclusion of matter source will not change the conclusion. Since caustics with lower codimensions are more difficult to bounce than those with higher codimensions [65], this result provides a strong support for the conjecture. We shall also argue that caustics with codimension higher than one will not form because of repulsive gravity generated by nonlinear higher curvature terms. Finally, we shall discuss implications to the recently proposed scenario of “dark matter as integration constant”.

The rest of this paper is organized as follows. In Sec. 2 we summarize basic equations in Hořava-Lifshitz gravity with the projectability condition. In Sec. 3 we shall argue that caustics do not form. Sec. 4 is devoted to discussion of implications to the “dark matter as integration constant” scenario.

⁴Here, it is assumed that the theory is renormalizable. It is also assumed that $\lambda = 1$ is a stable IR fixed point of RG flow. In this case, by reversing the direction of the RG flow and going towards the UV, λ should deviate from 1.

2 Basic equations

The basic quantities in the theory are the spatial metric g_{ij} , the shift vector N^i and the lapse function N . While g_{ij} and N^i can depend on both space and time coordinates, N can depend only on t . The fundamental symmetry of the theory is the invariance under the foliation-preserving diffeomorphism:

$$t \rightarrow \tilde{t}(t), \quad x^i \rightarrow \tilde{x}^i(t, x). \quad (2.1)$$

Under the infinitesimal transformation

$$\delta t = f(t), \quad \delta x^i = \zeta^i(t, x), \quad (2.2)$$

g_{ij} , N^i and N transform as

$$\begin{aligned} \delta g_{ij} &= f \partial_t g_{ij} + \mathcal{L}_\zeta g_{ij} \\ \delta N^i &= \partial_t (N^i f) + \partial_t \zeta^i + \mathcal{L}_\zeta N^i, \\ \delta(N_i) &= \partial_t (N_i f) + g_{ij} \partial_t \zeta^j + \mathcal{L}_\zeta N_i, \\ \delta N &= \partial_t (N f). \end{aligned} \quad (2.3)$$

Thus, N remains independent of spatial coordinates after transformation. In the IR, where dt and dx^i have the same scaling dimension, it makes sense to assemble g_{ij} , N^i and N into a 4-dimensional metric in the ADM form:

$$ds^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt)(dx^j + N^j dt). \quad (2.4)$$

The action is

$$I = I_g + I_m, \quad I_g = \frac{M_{Pl}^2}{2} \int dt dx^3 N \sqrt{g} (K^{ij} K_{ij} - \lambda K^2 + R + L_{z>1}), \quad (2.5)$$

where

$$K_{ij} = \frac{1}{2N} (\partial_t g_{ij} - D_i N_j - D_j N_i), \quad K = g^{ij} K_{ij}, \quad (2.6)$$

D_i is the covariant derivative compatible with g_{ij} , R is the Ricci scalar of g_{ij} , $L_{z>1}$ represents higher spatial curvature terms and I_m is the matter action. Here, we have rescaled the time coordinate so that the coefficients of $K^{ij} K_{ij}$ and R agree. The cosmological constant term will be included in matter action if necessary. Note that not only the gravitational action I_g but also the matter action I_m should be invariant under the foliation-preserving diffeomorphism.

By variation of the action with respect to $N(t)$, we obtain the Hamiltonian constraint

$$H_{g\perp} + H_{m\perp} = 0, \quad (2.7)$$

$$\begin{aligned}
H_{g\perp} &\equiv -\frac{\delta I_g}{\delta N} = \int dx^3 \sqrt{g} \mathcal{H}_{g\perp}, \quad \mathcal{H}_{g\perp} = \frac{M_{Pl}^2}{2} (K^{ij} p_{ij} - R - L_{z>1}), \\
H_{m\perp} &\equiv -\frac{\delta I_m}{\delta N} = \int dx^3 \sqrt{g} T_{\perp}^{\perp}, \quad T_{\perp}^{\perp} = T_{\mu\nu} n^{\mu} n^{\nu}.
\end{aligned} \tag{2.8}$$

Here, p_{ij} and n^{μ} are defined as

$$p_{ij} \equiv K_{ij} - \lambda K g_{ij}, \tag{2.9}$$

and

$$n_{\mu} dx^{\mu} = -N dt, \quad n^{\mu} \partial_{\mu} = \frac{1}{N} (\partial_t - N^i \partial_i). \tag{2.10}$$

Variation with respect to $N^i(t, x)$ leads to the momentum constraint

$$\mathcal{H}_{gi} + \mathcal{H}_{mi} = 0, \tag{2.11}$$

$$\begin{aligned}
\mathcal{H}_{gi} &\equiv -\frac{1}{\sqrt{g}} \frac{\delta I_g}{\delta N^i} = -M_{Pl}^2 D^j p_{ij}, \\
\mathcal{H}_{mi} &\equiv -\frac{1}{\sqrt{g}} \frac{\delta I_m}{\delta N^i} = T_{i\mu} n^{\mu}.
\end{aligned} \tag{2.12}$$

As in general relativity, the gravitational action can be written as the sum of kinetic terms and constraints up to boundary terms:

$$I_g = \int dt dx^3 [\pi^{ij} \partial_t g_{ij} - N^i \mathcal{H}_{gi}] - \int dt N H_{g\perp} + (\text{boundary terms}), \tag{2.13}$$

where π^{ij} is momentum conjugate to g_{ij} given by

$$\pi^{ij} \equiv \frac{\delta I_g}{\delta(\partial_t g_{ij})} = M_{Pl}^2 \sqrt{g} p^{ij}, \quad p^{ij} \equiv g^{ik} g^{jl} p_{kl}. \tag{2.14}$$

The Hamiltonian corresponding to the time t is the sum of constraints and boundary terms as

$$H_g[\partial_t] = N H_{g\perp} + \int dx^3 N^i \mathcal{H}_{gi} + (\text{boundary terms}). \tag{2.15}$$

Finally, by variation with respect to $g_{ij}(t, x)$, we obtain dynamical equation

$$\mathcal{E}_{gij} + \mathcal{E}_{mij} = 0, \tag{2.16}$$

$$\begin{aligned}
\mathcal{E}_{gij} &\equiv g_{ik} g_{jl} \frac{2}{N \sqrt{g}} \frac{\delta I_g}{\delta g_{kl}}, \\
\mathcal{E}_{mij} &\equiv g_{ik} g_{jl} \frac{2}{N \sqrt{g}} \frac{\delta I_m}{\delta g_{kl}} = T_{ij}.
\end{aligned} \tag{2.17}$$

The explicit expression for \mathcal{E}_{gij} is given by

$$\begin{aligned}\mathcal{E}_{gij} = & M_{Pl}^2 \left[-\frac{1}{N}(\partial_t - N^k D_k)p_{ij} - Kp_{ij} + 2K_i^k p_{kj} \right. \\ & \left. + \frac{1}{N}(p_{ik}D_j N^k + p_{jk}D_i N^k) + \frac{1}{2}g_{ij}K^{kl}p_{kl} - G_{ij} \right] + \mathcal{E}_{z>1ij},\end{aligned}\quad (2.18)$$

where $\mathcal{E}_{z>1ij}$ is the contribution from $L_{z>1}$ and G_{ij} is Einstein tensor of g_{ij} .

The invariance of I_α under the infinitesimal transformation (2.3) leads to the following conservation equations, where α represents g or m .

$$\begin{aligned}N\partial_t H_{\alpha\perp} + \int dx^3 \left[N^i \partial_t (\sqrt{g}\mathcal{H}_{\alpha i}) + \frac{1}{2}N\sqrt{g}\mathcal{E}_\alpha^{ij}\partial_t g_{ij} \right] &= 0, \\ \frac{1}{N}(\partial_t - N^j D_j)\mathcal{H}_{\alpha i} + K\mathcal{H}_{\alpha i} - \frac{1}{N}\mathcal{H}_{\alpha j}D_i N^j - D^j \mathcal{E}_{\alpha ij} &= 0.\end{aligned}\quad (2.19)$$

3 Absence of caustics

The vector n^μ defined in (2.10) has unit norm and is orthogonal to constant time hypersurfaces. Since the lapse function N depends only on time, n^μ follows the geodesic equation.

$$n^\mu \nabla_\mu n_\nu = n^\mu \nabla_\nu n_\mu = \frac{1}{2}\partial(n^\mu n_\mu) = 0. \quad (3.1)$$

In general relativity, a congruence of geodesics would generically form caustics. As an example, let us consider a congruence of geodesics orthogonal to the hypersurface $t = T(x)$ in Minkowski spacetime $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. By introducing a new time coordinate τ as the proper time along each geodesic and a new spatial coordinate X as the value of x at the intersection of each geodesic with the hypersurface $t = T(x)$, the Minkowski metric is rewritten as

$$ds^2 = -d\tau^2 + a^2(\tau, X)dX^2 + dy^2 + dz^2, \quad a(\tau, X) = \sqrt{1 - [T'(X)]^2} \left[1 - \frac{\tau}{\tau_c(X)} \right], \quad (3.2)$$

where

$$\tau_c(X) = \frac{[1 - T'(X)^2]^{3/2}}{-T''(X)}. \quad (3.3)$$

The metric component $a^2(\tau, X)$ vanishes at finite proper time $\tau = \tau_c(X)$ and, thus, the congruence of geodesics form caustics. In general relativity with a certain energy condition, it is easy to show that a congruence of geodesics forms caustics in more general situations essentially because gravity is attractive.

On the other hand, for the vector n^μ in the Hořava-Lifshitz gravity, higher curvature terms become important near (would-be) caustics and provide negative effective

energy and repulsive gravity. Also, λ should deviate from 1 by RG flow (see footnote 4) and, thus, the kinetic terms also contribute differently from general relativity.

3.1 Codimension higher than one

For codimension higher than one, the spatial curvature of the constant time hypersurface increases near (would-be) caustics. The system enters the non-relativistic regime and higher spatial curvature terms become important. Among them, highest order terms (e.g. curvature cubic terms in the case of $z = 3$) generate the strongest restoring force. As in some early universe models [4, 5, 9], we expect that the (would-be) caustics should bounce at short distance scales if codimension is higher than one. Note that this is not because of deviation from geodesics⁵ but because of repulsive gravity at short distances. In general relativity, congruence of geodesics would form caustics because gravity is attractive. On the other hand, for the vector n^μ in Hořava-Lifshitz gravity, higher curvature terms become important near the (would-be) caustics and provide repulsive gravity and thus bounce.

For an odd z , the sign of the highest nonlinear spatial curvature term can in principle change. Nonetheless, as far as a contracting region has a finite volume, the leading (would-be) divergence in the spatial curvature at late time is expected to be positive. For this reason, even with an odd z , we expect would-be caustics to bounce eventually. On the other hand, numerical confirmation of this kind of behavior probably requires a rather wide dynamic range since we have to wait until the finiteness of the contracting region becomes important. Thus, for numerical purposes it is probably easier to consider a large enough, even z . We hope to perform numerical analysis of the bouncing behavior in the future.

One might worry about the fact that the anisotropic scaling in the UV leads to the scaling $\propto 1/a^{z+3}$ for radiation energy density [20]. For $z = 3$, this might cause difficulties for bouncing cosmology with FRW (Friedmann-Robertson-Walker) spacetime since the radiation energy density scales in the same way as the $z = 3$ higher curvature terms and has the opposite sign. On the other hand, in the present situation, the contracting region has just a finite volume and thus radiation does not have to be comoving with the vector n^μ . Indeed, while the “dark matter” is pressureless and the vector n^μ follows geodesics, the large pressure ($P_{rad} = (z/3)\rho_{rad}$ in the UV) acts as extra repulsive force for radiation and prevents radiation from following

⁵ In the case of ghost condensate [64], the derivative of the scalar field responsible for the condensate deviates from geodesics because of higher derivative terms [65]. On the other hand, in Hořava-Lifshitz gravity the vector n^μ always satisfies the geodesic equation (3.1).

geodesics during inhomogeneous contraction. Thus, radiation can easily diffuse and grows much more slowly than the $z = 3$ curvature terms. For this reason, for $z = 3$, radiation does not cause difficulties. For $z > 3$, highest nonlinear curvature terms ($\propto 1/a^{2z}$) grows faster than radiation energy density even in FRW spacetime and, thus, radiation does not prevent the highest curvature terms from acting as restoring forces.

Therefore, unless the whole universe contracts, (would-be) caustics with codimension higher than one should bounce due to higher curvature terms. If the universe is completely homogeneous and flat, i.e. if the universe is a flat FRW spacetime, then higher spatial curvature terms vanish classically. Hence, a contracting flat FRW does not bounce classically. However, quantum mechanically, there must be fluctuations and nonlinear higher spatial curvature terms must acquire non-vanishing expectation values. Those fluctuations grow as the universe contracts. Therefore, if z is large enough and perhaps if z is even (so that the highest nonlinear spatial curvature terms always act as a restoring force) then a contracting flat FRW universe might also bounce after all. Further investigation of this issue is worthwhile.

Note that the bounce is provided by nonlinear terms. Therefore, if we analyzed behaviors of the vector n^μ and constant time hypersurfaces without including those nonlinear terms then we would not be able to see the bounce and would instead see caustics forming. It is likely that this is closely related to instabilities of linear perturbation found in [38]. It is also important to include backreactions of the higher spatial curvature terms to the geometry since, as already stated, the bounce is not due to deviation from geodesics but due to repulsive gravity at short distances. If we did not take into account backreactions to the geometry, one would simply conclude formation of caustics [55]. Without taking into account nonlinear terms and backreactions to the geometry, we would never be able to describe the system properly.

3.2 Codimension one

On the other hand, for codimension one, higher spatial curvature terms do not help. This is in accord with the observation in [65] that caustics with lower codimensions are more difficult to bounce than those with higher codimensions. In order to see it, let us consider the following ansatz with plane symmetry.

$$N = \alpha(t), \quad N^i \partial_i = \beta(t, x) \partial_x, \quad g_{ij} dx^i dx^j = a(t, x)^2 dx^2 + dy^2 + dz^2. \quad (3.4)$$

We have the foliation-preserving diffeomorphism

$$t \rightarrow \tilde{t}(t), \quad x \rightarrow \tilde{x}(t, x), \quad y \rightarrow y, \quad z \rightarrow z. \quad (3.5)$$

By using this symmetry, we can set $\alpha = 1$ and $a = 1$. The ansatz is now reduced to

$$N = 1, \quad N^i \partial_i = \beta(t, x) \partial_x, \quad g_{ij} dx^i dx^j = dx^2 + dy^2 + dz^2. \quad (3.6)$$

We still have the residual symmetry

$$t \rightarrow t + t_0, \quad x \rightarrow x + x_0(t), \quad (3.7)$$

where t_0 is a constant and x_0 is a function of t . It is now evident that higher spatial curvature terms do not help since the spatial metric is flat.

Actually, what prevents caustics with codimension one from forming is deviation of λ from 1. In order to recover general relativity in the IR, $\lambda = 1$ must be an IR fixed point of RG flow. However, near (would-be) caustics, the system enters the UV regime and λ should deviate from 1 by RG flow (see footnote 4). In the following, we shall show that there is no caustics with codimension one if $\lambda \neq 1$.

Ideally speaking, it is appropriate to take into account fully quantum mechanical effects to analyze behavior of the system near would-be caustics. However, at this stage where renormalizability (beyond power-counting) and RG flow of the theory are not yet understood, it is not easy to perform fully quantum mechanical analysis. For this reason, in this paper, we partially take into account quantum effects by including the scale-dependence of coupling constants (especially, deviation of λ from 1) in classical equations of motion.

As already argued in the previous subsection, radiation energy density does not become significant near (would-be) caustics simply because radiation does not have to be comoving with the vector n^μ and thus can diffuse. The same applies to other forms of matter fields. For this reason, in the following we ignore matter fields and set $T_{\mu\nu} = 0$.

For the ansatz (3.6), it is easy to show that

$$\begin{aligned} \mathcal{H}_{gx} &= -(\lambda - 1)M_{Pl}^2 \beta'', \\ \mathcal{E}_{gxx} &= (\lambda - 1)M_{Pl}^2 \left[-\dot{\beta}' + \beta\beta'' + \frac{1}{2}(\beta')^2 \right], \\ \mathcal{E}_{gyy} &= \mathcal{E}_{gzz} = \lambda M_{Pl}^2 \left[-\dot{\beta}' + \beta\beta'' + \frac{1}{2}(\beta')^2 \right] + \frac{M_{Pl}^2}{2}(\beta')^2, \end{aligned} \quad (3.8)$$

where an overdot and a prime denote derivatives with respect to t and x , respectively. With $\lambda \neq 1$, the general solution to the set of equations $\mathcal{H}_{gx} = \mathcal{E}_{gxx} = \mathcal{E}_{gyy} = \mathcal{E}_{gzz} = 0$ is

$$\beta = \beta_0(t), \quad (3.9)$$

where $\beta_0(t)$ is an arbitrary function of t ⁶. Thus, by using the residual symmetry (3.7) we can set

$$\beta = 0. \quad (3.10)$$

This shows that there is no caustics with codimension one.

4 Dark matter as integration constant

As already mentioned, in Hořava-Lifshitz gravity there is no local Hamiltonian constraint. This might cause worries since in general relativity the local Hamiltonian constraint is nothing but the Poisson equation, which is one of the most important equations for gravity. Actually, in the following we shall see that the absence of local Hamiltonian constraint leads to an interesting consequence. A component which behaves like dark matter emerges after solving the system of equations when we interpret general solutions. We will see that the Poisson equation with “dark matter” built-in is satisfied by the solutions.

Now, following ref. [41], let us define energy density ρ_d of “dark matter as integration constant” by

$$\rho_d \equiv -\frac{M_{Pl}^2}{2}(K^{ij}p_{ij} - R - L_{z>1}) - T_{\mu\nu}n^\mu n^\nu. \quad (4.1)$$

Consistency with (2.19) requires that

$$\partial_t \int dx^3 \sqrt{g} \rho_d = 0, \quad (4.2)$$

but this is automatically satisfied because of the Hamiltonian constraint. Equations of motion in Hořava-Lifshitz gravity are summarized as

$$M_{Pl}^2 \tilde{G}^{(4)\mu\nu} = T^{\mu\nu} + \rho_d n^\mu n^\nu, \quad (4.3)$$

$$\int dx^3 \sqrt{g} \rho_d = 0, \quad (4.4)$$

where

$$M_{Pl}^2 \tilde{G}^{(4)\mu\nu} = -\mathcal{H}_{g\perp} n^\mu n^\nu + \mathcal{H}_{gi} g^{ij} \left[n^\mu \left(\frac{\partial}{\partial x^j} \right)^\nu + n^\nu \left(\frac{\partial}{\partial x^j} \right)^\mu \right] - \mathcal{E}_{gij} g^{ik} g^{jl} \left(\frac{\partial}{\partial x^k} \right)^\mu \left(\frac{\partial}{\partial x^l} \right)^\nu. \quad (4.5)$$

⁶ With $\lambda = 1$, we would obtain $\beta = \beta_{IR} \equiv \beta_0(t) + x/(t_c - t)$, where $\beta_0(t)$ is an arbitrary function of t and t_c is a constant. By using the residual symmetry (3.7) we can set $\beta_{IR} = x/(t_c - t)$. This would diverge as $t \rightarrow t_c$ and thus represents a would-be caustics. However, as already stated, near a would-be caustics, the system enters the UV regime and λ should deviate from 1 by RG flow (see footnote 4). Thus, this solution is invalidated before actually reaching $t = t_c$.

By taking divergence of (4.3), we obtain

$$(\partial_{\perp}\rho_d + K\rho_d)n_{\mu} = -\nabla^{\nu}(T_{\mu\nu} - M_{Pl}^2\tilde{G}_{\mu\nu}^{(4)}), \quad (4.6)$$

where $\partial_{\perp} = n^{\mu}\partial_{\mu}$. Because of the spatial diffeomorphism invariance, the right hand side is proportional to n_{μ} . Thus, this equation has only one non-vanishing component. By contracting with n^{μ} , we obtain the (non-)conservation equation of “dark matter” [41]:

$$\partial_{\perp}\rho_d + K\rho_d = n^{\mu}\nabla^{\nu}[T_{\mu\nu} - M_{Pl}^2\tilde{G}_{\mu\nu}^{(4)}]. \quad (4.7)$$

In the IR limit with $\lambda \rightarrow 1$, $\tilde{G}^{(4)\mu\nu}$ reduces to the 4-dimensional Einstein tensor $G^{(4)\mu\nu}$ and eq. (4.3) reduces to the Einstein equation with “dark matter”

$$M_{Pl}^2 G^{(4)\mu\nu} = T^{\mu\nu} + \rho_d n^{\mu} n^{\nu}. \quad (4.8)$$

From this equation, one can obtain the Poisson equation with “dark matter” built-in. In the same limit, (4.7) reduces to the conservation equation,

$$\partial_{\perp}\rho_d + K\rho_d = 0. \quad (4.9)$$

Note that ρ_d can be positive everywhere in our patch of the universe. In a homogeneous spacetime such as the FRW spacetime, the global Hamiltonian constraint is as good as local one since all spatial points are equivalent. However, in inhomogeneous spacetimes there are drastic differences. If the whole universe is much larger than the present Hubble volume then it is natural to expect that the universe far beyond the present Hubble horizon is different from our patch of the universe inside the horizon. In this case, the global Hamiltonian constraint (4.4) does not restrict the universe inside the horizon. Even if we approximate our patch of the universe inside the present horizon by a FRW spacetime, the whole universe can include inhomogeneities of super-horizon scales and, thus, the global Hamiltonian constraint does not restrict the FRW spacetime which just approximates the behavior inside the horizon. Therefore, ρ_d can be positive everywhere in our patch of the universe inside the present Hubble horizon.

In the UV epoch, fluctuations of matter fields as well as metric fluctuations act as the source term in (4.7). Since such fluctuations include modes with various wavelengths, fluctuations of “dark matter” are generated with various wavelengths, including those far longer than the size corresponding to the present Hubble horizon. Therefore, there must certainly be large enough regions with positive ρ_d . In principle it should be possible to predict a typical amplitude of ρ_d , once a model of the early universe is specified in the context of Hořava-Lifshitz gravity.

Note also that we do not have to promote the “dark matter as integration constant” to an independent dynamical field as far as the scalar graviton, the tensor graviton and matter fields are considered as independent dynamical fields in the initial value formulation. The initial value formulation of Hořava-Lifshitz gravity consists of dynamical equation (2.16), global Hamiltonian constraint (2.7), local momentum constraint (2.11) and gauge conditions. Of course, the constraints are preserved by the dynamical equation. In this language, the “dark matter” emerges only after solving the system of equations when we try to interpret a solution.

In order to describe the scalar graviton, the commonly used method called Stückelberg formalism does not seem to be useful here. If we adopted it to construct an effective field theory of the scalar graviton then, unlike ghost condensate [64], the foliation-preserving diffeomorphism would forbid h_{00}^2 and thus $\dot{\pi}^2$. For this reason, this description would not include a healthy kinetic term even with $\lambda \neq 1$ and would get strongly coupled at all scales in Minkowski background [55]. In non-vanishing “dark matter” backgrounds, the strong-coupling scale of the Stückelberg field becomes finite but is still as low as $\rho_d^{1/4}$ even with $\lambda \neq 1$ [55]. This indicates breakdown of this description but does not imply inconsistency of the underlining UV theory. The physical reason for this is that the Stückelberg field has vanishing overlap with the scalar graviton. Indeed, if we adopt for example the gauge used in [1] (without introducing a Stückelberg field) then the scalar graviton has a finite kinetic term away from $\lambda = 1$ and does not exhibit the strong coupling mentioned above for the Stückelberg field. (If we introduce the Stückelberg field then we should use a gauge-invariant variable representing the scalar graviton and solve constraint equations. After all, we should be able to obtain a finite kinetic term away from $\lambda = 1$ in Minkowski background.) Of course, even in this description, we have to take into account nonlinear interactions carefully when we take the limit $\lambda \rightarrow 1$ to see if there is an analogue of the Vainshtein effect [63]. We hope to come back to the issue of Vainshtein-like effect in the near future.

If we consider a group of many microscopic lumps of “dark matter” then collisions and bounces among those lumps may accumulate to generate non-trivial effects in macroscopic scales. This is exactly the spirit of renormalization group. As far as gravity at astrophysical scales is concerned, this kind of collective behavior of small lumps of “dark matter” might mimic behavior of a cluster of particles with velocity dispersion. If this is the case then microscopic lumps of “dark matter” play the role of particles in usual dark matter models. It is certainly interesting to see what happens if a group of astrophysically large number of such microscopic lumps of “dark matter” collides with another group. Clearly, detailed investigation is necessary to understand

rich dynamics of “dark matter” from microscopic to macroscopic scales.

The foliation-preserving diffeomorphism invariance is the fundamental symmetry of the theory. Thus, the whole system including all matter fields must respect this symmetry. In particular, the matter action must be invariant under the 3-dimensional spatial diffeomorphism. This fact, combined with the orthogonality of the flow of “dark matter” to the constant time hypersurface, means that the dispersion relation for each matter field,

$$\omega^2 = \frac{1}{M^{2z-2}} k^{2z} + \cdots + c_s^2 k^2 + m^2, \quad (4.10)$$

is defined in the rest frame of “dark matter”. Thus, in the region where matter fields move relative to “dark matter” with large relative velocities, higher order terms in the dispersion relation of matter fields can become important. It is certainly interesting to investigate astrophysical implications of such effects.

Note added

The issue of caustics in Hořava-Lifshitz gravity with the projectability condition was discussed in [55]. Following recommendation by an anonymous referee, here we would like to comment on it. Ref. [55] indeed has three statements about Hořava-Lifshitz gravity with the projectability condition: (i) “dark matter” forms caustics; (ii) “dark matter” is described by ghost condensate [64]; (iii) the scalar sector gets strongly coupled at the scale $\Lambda \sim \rho_d^{1/4}$ even with $\lambda \neq 1$. Actually, these three comments are not correct for the following reasons. (i) They did not take into account repulsive gravity due to nonlinear higher curvature terms as mentioned in Sec. 3 of the present paper. (ii) Ghost condensate and Hořava-Lifshitz gravity have different symmetries as mentioned in the third-to-the-last paragraph of Sec. 4 of the present paper. (iii) The strong coupling away from $\lambda = 1$ found in [55] indicates sickness of their description, i.e. the way dynamical degrees of freedom are identified, but does not imply inconsistency of the underlining UV theory. For this point, see the fourth-to-the-last and the third-to-the-last paragraphs of Sec. 4 of the present paper.

Acknowledgements

The author would like to thank Diego Blas, Andrei Frolov, Oriol Pujolas, Marco Serone, Sergey Sibiryakov, Takahiro Tanaka and Chul-Moon Yoo for useful discussions. The work of the author was supported in part by MEXT through a Grant-in-Aid for Young Scientists (B) No. 17740134, by JSPS through a Grant-in-Aid for

Creative Scientific Research No. 19GS0219, and by the Mitsubishi Foundation. This work was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

References

- [1] P. Horava, “Quantum Gravity at a Lifshitz Point,” arXiv:0901.3775 [hep-th].
- [2] P. Horava, “Membranes at Quantum Criticality,” JHEP **0903**, 020 (2009) [arXiv:0812.4287 [hep-th]].
- [3] T. Takahashi and J. Soda, “Chiral Primordial Gravitational Waves from a Lifshitz Point,” arXiv:0904.0554 [hep-th].
- [4] G. Calcagni, “Cosmology of the Lifshitz universe,” arXiv:0904.0829 [hep-th].
- [5] E. Kiritsis and G. Kofinas, “Horava-Lifshitz Cosmology,” arXiv:0904.1334 [hep-th].
- [6] J. Kluson, “Branes at Quantum Criticality,” arXiv:0904.1343 [hep-th].
- [7] H. Lu, J. Mei and C. N. Pope, “Solutions to Horava Gravity,” arXiv:0904.1595 [hep-th].
- [8] S. Mukohyama, “Scale-invariant cosmological perturbations from Horava-Lifshitz gravity without inflation,” JCAP **0906**, 001 (2009) [arXiv:0904.2190 [hep-th]].
- [9] R. Brandenberger, “Matter Bounce in Horava-Lifshitz Cosmology,” arXiv:0904.2835 [hep-th].
- [10] H. Nikolic, “Horava-Lifshitz gravity, absolute time, and objective particles in curved space,” arXiv:0904.3412 [hep-th].
- [11] K. I. Izawa, “A Note on Quasi-Riemannian Gravity with Higher Derivatives,” arXiv:0904.3593 [hep-th].
- [12] H. Nastase, “On IR solutions in Horava gravity theories,” arXiv:0904.3604 [hep-th].
- [13] R. G. Cai, L. M. Cao and N. Ohta, “Topological Black Holes in Horava-Lifshitz Gravity,” arXiv:0904.3670 [hep-th].

- [14] R. G. Cai, Y. Liu and Y. W. Sun, “On the $z=4$ Horava-Lifshitz Gravity,” arXiv:0904.4104 [hep-th].
- [15] Y. S. Piao, “Primordial Perturbation in Horava-Lifshitz Cosmology,” arXiv:0904.4117 [hep-th].
- [16] X. Gao, “Cosmological Perturbations and Non-Gaussianities in Hořava-Lifshitz Gravity,” arXiv:0904.4187 [hep-th].
- [17] E. O. Colgain and H. Yavartanoo, “Dyonic solution of Horava-Lifshitz Gravity,” arXiv:0904.4357 [hep-th].
- [18] T. Sotiriou, M. Visser and S. Weinfurtner, “Phenomenologically viable Lorentz-violating quantum gravity,” arXiv:0904.4464 [hep-th].
- [19] B. Chen and Q. G. Huang, “Field Theory at a Lifshitz Point,” arXiv:0904.4565 [hep-th].
- [20] S. Mukohyama, K. Nakayama, F. Takahashi and S. Yokoyama, “Phenomenological Aspects of Horava-Lifshitz Cosmology,” arXiv:0905.0055 [hep-th].
- [21] Y. S. Myung and Y. W. Kim, “Thermodynamics of Hořava-Lifshitz black holes,” arXiv:0905.0179 [hep-th].
- [22] R. G. Cai, B. Hu and H. B. Zhang, “Dynamical Scalar Degree of Freedom in Horava-Lifshitz Gravity,” arXiv:0905.0255 [hep-th].
- [23] D. Orlando and S. Reffert, “On the Renormalizability of Horava-Lifshitz-type Gravities,” arXiv:0905.0301 [hep-th].
- [24] C. Gao, “Modified gravity in Arnowitt-Deser-Misner formalism,” arXiv:0905.0310 [astro-ph.CO].
- [25] T. Nishioka, “Horava-Lifshitz Holography,” arXiv:0905.0473 [hep-th].
- [26] A. Kehagias and K. Sfetsos, “The black hole and FRW geometries of non-relativistic gravity,” arXiv:0905.0477 [hep-th].
- [27] S. K. Rama, “Anisotropic Cosmology and (Super)Stiff Matter in Hořava’s Gravity Theory,” arXiv:0905.0700 [hep-th].
- [28] R. G. Cai, L. M. Cao and N. Ohta, “Thermodynamics of Black Holes in Horava-Lifshitz Gravity,” arXiv:0905.0751 [hep-th].

- [29] A. Ghodsi, “Toroidal solutions in Horava Gravity,” arXiv:0905.0836 [hep-th].
- [30] Y. S. Myung, “Thermodynamics of black holes in the deformed Hořava-Lifshitz gravity,” arXiv:0905.0957 [hep-th].
- [31] S. Chen and J. Jing, “Quasinormal modes of a black hole in the deformed Hořava-Lifshitz gravity,” arXiv:0905.1409 [gr-qc].
- [32] J. Kluson, “Stable and Unstable D-Branes at Criticality,” arXiv:0905.1483 [hep-th].
- [33] R. A. Konoplya, “Towards constraining of the Horava-Lifshitz gravities,” arXiv:0905.1523 [hep-th].
- [34] S. Chen and J. Jing, “Strong field gravitational lensing in the deformed Hořava-Lifshitz black hole,” arXiv:0905.2055 [gr-qc].
- [35] B. Chen, S. Pi and J. Z. Tang, “Scale Invariant Power Spectrum in Hořava-Lifshitz Cosmology without Matter,” arXiv:0905.2300 [hep-th].
- [36] C. Charmousis, G. Niz, A. Padilla and P. M. Saffin, “Strong coupling in Horava gravity,” arXiv:0905.2579 [hep-th].
- [37] M. Li and Y. Pang, “A Trouble with Hořava-Lifshitz Gravity,” arXiv:0905.2751 [hep-th].
- [38] T. P. Sotiriou, M. Visser and S. Weinfurtner, “Quantum gravity without Lorentz invariance,” arXiv:0905.2798 [hep-th].
- [39] Y. W. Kim, H. W. Lee and Y. S. Myung, “Nonpropagation of scalar in the deformed Hořava-Lifshitz gravity,” arXiv:0905.3423 [hep-th].
- [40] E. N. Saridakis, “Horava-Lifshitz Dark Energy,” arXiv:0905.3532 [hep-th].
- [41] S. Mukohyama, “Dark matter as integration constant in Horava-Lifshitz gravity,” arXiv:0905.3563 [hep-th].
- [42] G. Calcagni, “Detailed balance in Horava-Lifshitz gravity,” arXiv:0905.3740 [hep-th].
- [43] X. Gao, Y. Wang, R. Brandenberger and A. Riotto, “Cosmological Perturbations in Hořava-Lifshitz Gravity,” arXiv:0905.3821 [hep-th].

- [44] M. Minamitsuji, “Classification of cosmology with arbitrary matter in the Hořava-Lifshitz theory,” arXiv:0905.3892 [astro-ph.CO].
- [45] A. Wang and Y. Wu, “Thermodynamics and classification of cosmological models in the Horava-Lifshitz theory of gravity,” arXiv:0905.4117 [hep-th].
- [46] A. A. Kocharyan, “Is nonrelativistic gravity possible?,” arXiv:0905.4204 [hep-th].
- [47] S. Nojiri and S. D. Odintsov, “Covariant Horava-like renormalizable gravity and its FRW cosmology,” arXiv:0905.4213 [hep-th].
- [48] M. Sakamoto, “Strong Coupling Quantum Einstein Gravity at a $z=2$ Lifshitz Point,” arXiv:0905.4326 [hep-th].
- [49] M. i. Park, “The Black Hole and Cosmological Solutions in IR modified Horava Gravity,” arXiv:0905.4480 [hep-th].
- [50] M. Botta-Cantcheff, N. Grandi and M. Sturla, “Wormhole solutions to Horava gravity,” arXiv:0906.0582 [hep-th].
- [51] Y. S. Myung, “Propagations of massive graviton in the deformed Hořava-Lifshitz gravity,” arXiv:0906.0848 [hep-th].
- [52] C. Germani, A. Kehagias and K. Sfetsos, “Relativistic Quantum Gravity at a Lifshitz Point,” arXiv:0906.1201 [hep-th].
- [53] A. Ghodsi and E. Hatefi, “Extremal rotating solutions in Horava Gravity,” arXiv:0906.1237 [hep-th].
- [54] Y. F. Cai and E. N. Saridakis, “Non-singular cosmology in a model of non-relativistic gravity,” arXiv:0906.1789 [hep-th].
- [55] D. Blas, O. Pujolas and S. Sibiryakov, “On the Extra Mode and Inconsistency of Horava Gravity,” arXiv:0906.3046 [hep-th].
- [56] Y. F. Cai and X. Zhang, “Primordial perturbation with a modified dispersion relation,” arXiv:0906.3341 [astro-ph.CO].
- [57] R. Iengo, J. G. Russo and M. Serone, “Renormalization group in Lifshitz-type theories,” arXiv:0906.3477 [hep-th].
- [58] Y. Kawamura, “Misleading Coupling Unification and Lifshitz Type Gauge Theory,” arXiv:0906.3773 [hep-ph].

- [59] M. i. Park, “A Test of Horava Gravity: The Dark Energy,” arXiv:0906.4275 [hep-th].
- [60] S. R. Coleman and S. L. Glashow, “High-Energy Tests of Lorentz Invariance,” Phys. Rev. D **59**, 116008 (1999) [arXiv:hep-ph/9812418].
- [61] T. Jacobson, S. Liberati and D. Mattingly, “TeV astrophysics constraints on Planck scale Lorentz violation,” Phys. Rev. D **66**, 081302 (2002) [arXiv:hep-ph/0112207].
- [62] G. D. Moore and A. E. Nelson, “Lower bound on the propagation speed of gravity from gravitational Cherenkov radiation,” JHEP **0109**, 023 (2001) [arXiv:hep-ph/0106220].
- [63] A. I. Vainshtein, “To the problem of nonvanishing gravitation mass,” Phys. Lett. B **39**, 393 (1972).
- [64] N. Arkani-Hamed, H. C. Cheng, M. A. Luty and S. Mukohyama, “Ghost condensation and a consistent infrared modification of gravity,” JHEP **0405**, 074 (2004) [arXiv:hep-th/0312099].
- [65] N. Arkani-Hamed, H. C. Cheng, M. A. Luty, S. Mukohyama and T. Wiseman, “Dynamics of Gravity in a Higgs Phase,” JHEP **0701**, 036 (2007) [arXiv:hep-ph/0507120].